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Effective Potential for Scalar QED in (2 + 1) Dimensions

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We calculate the renormalized one loop approximation to the effective potential for scalar electrodynamics in (2 + 1) dimensions. We study its gauge dependence and show that it satisfies the renormalization group equation since it is independent on any renormalization scale. This relies on the fact that we do not need to improve the effective potential in (2 + 1) dimensions.

1. INTRODUCTION

In general the study of lower dimensional situations is motivated by the fact that in the simpler setting we can learn useful things which can be applied to four-dimensional problems. Moreover, there are possible physical applications: The high-temperature behavior of four-dimensional field theories is governed by their three-dimensional analysis (Gross *et al.*, 1981; Dese *et al.*, 1982). Interesting condensed matter phenomena such as the quantum Hall effect and high-T_c supercondictivity appear to involve planer gauge-theoretic dynamics. Introducing anyons in (2 + 1)-dimensional physics created a new point of contact between solid-state physics and particle physics. Anyonic supercondictivity presents a candidate for the explanation of the superconditing properties observed in certain materials at high temperature.

The effective potential for a field theory as introduced by Euler, Heisenberg, and Schwinger is very useful in the study of spontaneous symmetry breaking. Unfortunately, an exact computation of the effective potential involves an infinite number of Feynman diagrams (Coleman and Wienberg, 1973). This is a difficult task, especially when several interactions are present.

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So we are only able to calculate the effective potential approximated to a few-loop approximation.

The expansion of the effective potential in four dimensions contains a renormalization scale which is arbitrary. The effect of its change can be absorbed into the changes in the coupling constant and the field. This problem arises from the fact that the effective potential does not satisfy the renormalization group equation. The renormalization group equation for the effective potential states that

$$\frac{dV}{dM} = 0$$

Then

$$\left[M \frac{\partial}{\partial M} + \beta(g_i) \frac{\partial}{\partial g_i} - \gamma\phi \frac{\partial}{\partial \phi}\right] V = 0$$

where $\beta = M\, dg_i/dM$ and γ is the anomalous dimension, $\gamma = -d\phi/dM$. Many procedures to improve the effective potential so as to satisfy the renormalization group equation have been suggested (Mckeon, 1984; Kastening, 1991; Bando *et al.*, 1992).

In this note, we calculate the renormalized one-loop approximation to the effective potential for scalar electrodynamics in (2 + 1) dimensions. This expression of the effective potential is required for studying the spontaneous symmetry breaking in (2 + 1) dimensions.

We study the gauge dependence of the effective potential; this dependence presents a difficulty in using the effective potential, since it may create false minima. We show explicitly this gauge dependence of the effective potential in (2 + 1) dimensions.

We note that the effective potential is independent on any renormalization scale, in contrast with the effective potential in four dimensions, where $V(\phi) \approx \phi^4[\ln(\phi^2/M^2) - \frac{1}{2}]$ and M is the renormalization scale. So in (2 + 1) dimensions we do not need to introduce an improvement to the effective potential, since it already satisfies the renormalization group equation.

2. EFFECTIVE POTENTIAL FOR SCALAR QED IN 2 + 1 DIMENSIONS

In scalar massless quantum electrodynamics,

$$S(\phi, A) = \int d^3x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + |(\partial_\mu + ieA_\mu)\phi|^2 - \frac{\lambda}{4}(|\phi|^2)^2\right] \quad (1)$$

where A_μ is the photon field with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and ϕ is the complex field

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

where ϕ_1, ϕ_2 are real. $\Gamma(\phi)$ is the generating function for connected, single-particle irreducible Green's functions of the charged fields. In general we can write $\Gamma(\phi)$ as the space-time integral of an effective Lagrangian

$$L_{\text{eff}} = -V(\phi) + \frac{1}{2} Z(\phi)\partial_\mu\phi\partial^\mu\phi + \cdots \tag{2}$$

The function $V(\phi)$, which does not contain any derivatives of ϕ, is called the effective potential. So for a constant field ϕ_c

$$\Gamma(\phi_c) = -\int d^3x\, V(\phi_c) \tag{3}$$

Then up to one loop we find

$$\int d^3x\, V_0(\phi) = -S(\phi)\,|_{\phi=\text{const}} \tag{4}$$

$$\int d^3x\, V_1(\phi) = -\frac{1}{2} \ln \operatorname{Det} G(\phi)\,|_{\phi=\text{const}} \tag{5}$$

where

$$G^{ij}S_{,jk} = -\delta^i_k$$

and

$$S_{,jk} = \left.\frac{\delta^2 S}{\delta\phi_j\delta\phi_k}\right|_{\phi=\phi_c=\text{const}}$$

Using this formula to calculate $V(\phi)$, where in this case $\phi = (\phi_1, \phi_2, A_\mu)$, we find that

$$V_0(\phi) = -S(\phi, 0) = \frac{\lambda}{4} (|\phi|^2)^2 \tag{6}$$

In order to evaluate $V_1(\phi)$, we need to calculate $G(\phi)$, which is given by

$$(G^{-1})_{ij} = -S_{,ij}$$

so we can write the Lagrangian of equation (1) in the form

$$L = (\phi^{+} A_{\mu})(G^{-1}(k))\begin{pmatrix} \phi \\ A_{\nu} \end{pmatrix}$$

where

$$G^{-1}(k) = \begin{pmatrix} k^2 - \frac{1}{4}\lambda|\phi|^2 & ek_{\mu}\phi \\ -e\phi^{+}k_{\nu} & g_{\mu\nu}(k^2 - e^2|\phi|^2) + (1/\alpha - 1)k_{\mu}k_{\nu} \end{pmatrix} \tag{7}$$

We evaluate the determinant of the $G^{-1}(k)$ matrix (which is a covariant quantity) by assuming a simple frame such as $k_1 = k_2 = 0$, $k_0 \neq 0$. In this frame we have $k^2 = k_0^2$ and the value of the determinant in this case is given by

$$\det G^{-1}(k) =$$

$$\det\begin{pmatrix} k_0^2 - \frac{\lambda}{4}|\phi|^2 & ek_0\phi & 0 & 0 \\ -e\phi^{+}k_0 & (k_0^2 - e^2|\phi|^2) + (\frac{1}{\alpha} - 1)k_0^2 & 0 & 0 \\ 0 & 0 & k_0^2 - e^2|\phi|^2 & 0 \\ 0 & 0 & 0 & k_0^2 - e^2|\phi|^2 \end{pmatrix} \tag{8}$$

so that

$$\det G^{-1}(k) =$$

$$(k_0^2 - e^2|\phi|^2)^2\left[\left(k_0^2 - \frac{\lambda}{4}|\phi|^2\right)\left(\left(k_0^2 - e^2|\phi|^2 + \left(\frac{1}{\alpha} - 1\right)k_0^2\right) + e^2|\phi|^2k_0^2\right]$$

As we mentioned, $\det G^{-1}(k)$ is invariant. Then in general we have

$$\det G^{-1}(k) =$$

$$(k^2 - e^2|\phi|^2)^2\left\{\left(k^2 - \frac{\lambda}{4}|\phi|^2\right)\left[k^2 - e^2|\phi|^2 + \left(\frac{1}{\alpha} - 1\right)k^2\right] + e^2|\phi|^2k^2\right\} \tag{9}$$

where α is the gauge fixing. The quantity α was kept here in order to investigate how the effective potential depends on the gauge fixing. Using (5), we get

$$V_1(\phi) = \int \frac{d^3k}{(2\pi)^3}\left[2\ln(k^2 - e^2|\phi|^2)\right.$$

$$+ \ln\left[\left(k^2 - \frac{\lambda}{4}|\phi|^2\right)\left[(k^2 - e^2|\phi|^2) + \left(\frac{1}{\alpha} - 1\right)k^2\right] - e^2k^2|\phi \tag{10}$$

Equation (10) can be written as

$$V_1(\phi) = \int \frac{d^3k}{(2\pi)^3} \left[2 \ln(k^2 - e^2|\phi|^2) + \ln\left(\frac{1}{\alpha} k^2 - A\right) + \ln(k^2 - B)\right]$$

where

$$AB = \frac{\lambda}{4} e^2|\phi|^4 \qquad \text{and} \qquad A + B = \frac{1}{\alpha}\frac{\lambda}{4}|\phi|^2$$

The integral is ultraviolet divergent. To evaluate it, we cut off the integral at $k^2 = \Lambda^2$, and we find

$$V_1(\phi) = \frac{1}{3\pi} [2(e^2|\phi|^2)^{3/2} + \alpha\sqrt{\alpha}A^{3/2} + B^{3/2}] \tag{11}$$

where we ignored any term that vanishes as Λ goes to infinity and we used a usual mass renormalization counterterm, which is determined by imposing the definition of the renormalized mass. Having renormalized the mass to vanish (i.e., $d^2V/d\phi^2 = 0$), we can make use of the determination of this counterterm which canceled the divergent part in V_1.

From equation (11) we note that $V(\phi)$ does not depend on any renormalization scale, in contrast with the effective potential in four dimensions. So the effective potential of the scalar QED in (2 + 1) dimensions automatically satisfies the renormalization group equation.

We conclude that in the current study, no further improvement is needed to the effective potential in (2 + 1) dimensions.

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